CHAPTER 1

On Constructing Distance and Similarity Measures based on Fuzzy Implications

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This chapter deals with the construction of distance and similarity measures by utilizing the theoretical advantages of the fuzzy implications. To this end the basic definitions of fuzzy implications are initially discussed and the conditions of typical distance and similarity measures that need to be satisfied are defined next. On the basis of this theory a straightforward methodology for building fuzzy implications based measures is analysed. The main advantage of the proposed methodology is its generality that makes it easy to be adopted in several types of fuzzy sets.

Keywords - Fuzzy implications, fuzzy similarity, fuzzy sets, intuitionistic fuzzy sets.

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1.1 Introduction

Various distance and similarity measures for fuzzy sets have been proposed in the literature e.g. [7, 15, 20, 25, 34, 43, 47, 49, 50] and [10, 11, 13, 27, 28, 29, 30, 31, 32, 33], respectively. The usefulness of distance and similarity measures for fuzzy implications lies in the fact of the non-uniqueness of classical implication when it is extended to fuzzy logic. So, in applications, one often must choose specific fuzzy operators to best match available data or experience and, to this end, similarity classifications are a useful tool. Finding the degree of similarity between the available fuzzy implications to a specific problem, helps us to calculate the degree of similarity of their behavior. In general, it is a difficult problem to select an appropriate fuzzy implication for approximate reasoning under each particular situation.

Quantifying similarity between fuzzy sets is very important in applications ranging from pattern recognition and machine learning, to decision making, market prediction, etc. [7].

A theoretical usefulness of similarity measures is that it enables us to classify fuzzy implications into equivalence classes, based on their degree of similarity. Also, the existence of an ordering relation between fuzzy implications helps us to order fuzzy implications in a more powerful and/or more functional classes. All measures listed here are intended for applications and may be customized according to the needs and intuition of the user.

We refer to two families of distance measures between binary fuzzy operators, along with its dual families of similarity measures, which have been proposed for this purpose.

One of these families, refers to the family of normalized distance measures between binary fuzzy operators, along with its dual family of similarity measures. These measures are based on matrix norms and arise from the study of the aggregate plausibility of set-operations [7].

The other of these families, based on the criteria, which used to select the most appropriate fuzzy implication, coming to the condition that the generalized modus tollens coincides with the classic modus tollens and the generalized modus ponens coincides with the classical modus ponens [23, 24, 42].

1.2 Preliminaries

In this section, we review standard notations and definitions from the literature of fuzzy sets, fuzzy implications, intuitionistic fuzzy sets and measures [15, 16, 17, 19, 35, 41, 46].

1.2.1 Fuzzy Implications. Basic Notations and Definitions

Let X denote a universe of discourse. Then, a fuzzy set A in X is defined as a set of ordered pairs $A = \{\langle x, \mu_A(x)\rangle | x \in X\}$, where the function $\mu_A(x) : X \to [0,1]$ defines the degree of membership of the element $x \in X$.

A binary operation i on the unit interval (i.e. a $[0,1] \times [0,1] \to [0,1]$ mapping) is called a *fuzzy intersection*, if it is an extension of the classical Boolean intersection:$i\left(a,b\right)\in\left[0,1\right],\forall a,b\in\left[0,1\right]$ and i (0,0) $=i\left(0,1\right)=i\left(1,0\right)=$ 0, i (1,1) $=$ 1.

A canonical model of fuzzy intersections is the family of triangular norms (briefly t-norms). A *t-norm* T is a function of the form: $T:\left[0,1\right]\times\left[0,1\right]\rightarrow\left[0,1\right]$, which is commutative, associative, non-decreasing, and $T\left(a,1\right)=a,\forall a\in\left[0,1\right]$.

A t-norm T is called *Archimedean*, if it is continuous and for $a\in\left(0,1\right),T\left(a,a\right)<a$, *nilpotent* if it is continuous and if, for all $a\in\left(0,1\right)$, there is $\nu\in\mathbb{N}$ such that $T(\overbrace{a,...,a}^{\nu})=0$. Archimedean norms have two forms: nilpotent ones and those which are not nilpotent. Those which are not nilpotent are called *strict*.

A function $n:\left[0,1\right]\rightarrow\left[0,1\right]$ is called a *negation*, if it is non-increasing, i.e. if $a\geq b$ then $n\left(a\right)\leq n\left(b\right)),\forall a,b\in\left[0,1\right]$ and $n\left(0\right)=1$, $n\left(1\right)=0$.

A negation n is called *strict*, if and only if n is continuous and strictly decreasing, i.e. if $a>b$ then $n\left(a\right)<n\left(b\right),\forall a,b\in\left[0,1\right]$.

A strict negation n is called *strong*, if it is self-inverse, i.e. $n\left(n\left(a\right)\right)=a$, $\forall a\in\left[0,1\right]$.

A function $S:\left[0,1\right]\times\left[0,1\right]\rightarrow\left[0,1\right]$ is called a *triangular conorm*, (briefly *t-conorm*) if satisfying the properties:

(i) $S\left(a,0\right)=a$ for all $a\in\left[0,1\right]$, (*Boundary Condition*).

(ii) if $a\leq$ c and $b\leq d$ then $S\left(a,b\right)\leq S\left(c,d\right)$, (*Monotonicity*).

(iii) $S\left(a,b\right)=S\left(b,a\right)$for all $a,b\in\left[0,1\right]$, (*Commutativity*).

(iv) $S\left(S\left(a,b\right),c\right)=S\left(a,S\left(b,c\right)\right)$for all $a,b,c\in\left[0,1\right]$, (*Associativity*).

A *fuzzy implication* I is a function of the form: $I:\left[0,1\right]\times\left[0,1\right]\rightarrow\left[0,1\right]$, which for any possible truth values a, b of given fuzzy propositions P, Q, respectively, defines the truth value, $I\left(a,b\right)$, of the conditional proposition "if P then Q". The function I should be an extension of the classical implication from the domain $\{0,1\}$ to the domain $[0,1]$, of truth-values in fuzzy logic.

The *implication operator* of classical logic is a mapping: $m:\{0,1\}\times\{0,1\}\rightarrow\{0,1\}$, which satisfies the conditions:

$$m\left(0,0\right)=m\left(0,1\right)=m\left(1,1\right)=1\,\text{and}\,m\left(1,0\right)=0.$$

The latter conditions are typically the minimum requirements for a fuzzy implication operator. In other words, fuzzy implications are required to reduce to the classical implication when truth-values are restricted to 0 and 1; i.e.

$$I\left(0,0\right)=I\left(0,1\right)=I\left(1,1\right)=1\,\text{and}\,I\left(1,0\right)=0.$$

One way of defining an implication operator m in classical logic is using formula

$$m\left(a,b\right)=\neg a\vee b,a,b\in\{0,1\}\qquad(1.1)$$

where $\neg a$ denotes the negation of a.

Another three different Boolean expressions for the implication function m are shown next:

$$m\left(a,b\right)=\max\left\{x\in\{0,1\}:a\wedge x\leq b\right\},a,b\in\{0,1\},\qquad(1.2)$$

$$m\left(a,b\right)=\neg a\vee\left(a\wedge b\right), \tag{1.3}$$

$$m\left(a,b\right)=\left(\neg a\wedge\neg b\right)\vee b,\forall a,b\in\left\{0,1\right\}. \tag{1.4}$$

Fuzzy logic extensions of the previous formulas respectively, are ($\forall a,b\in[0,1]$):

$$I_S\left(a,b\right)=S\left(n\left(a\right),b\right), \tag{1.5}$$

$$I_R\left(a,b\right)=\sup\left\{x\in[0,1]\mid T\left(a,x\right)\leq b\right\}, \tag{1.6}$$

$$I_{QL}\left(a,b\right)=S\left(n\left(a\right),T\left(a,b\right)\right), \tag{1.7}$$

$$I_D\left(a,b\right)=S\left(T\left(n\left(a\right),n\left(b\right)\right),b\right), \tag{1.8}$$

where S, T and n denote a t-conorm, a t-norm, and a strong fuzzy negation, respectively. Note that functions S and T are dual (with respect to n). Recall that a t-norm T and a t-conorm S are called *dual* (with respect to a fuzzy negation n) if and only if both $S\left(n\left(a\right),n\left(b\right)\right)=n\left(T\left(n\left(a\right),n\left(b\right)\right)\right)$and $T\left(n\left(a\right),n\left(b\right)\right)=n\left(S\left(a,b\right)\right)$ hold $\forall a,b\in[0,1]$. A triple $\langle S,T,n\rangle$ where n is a strong negation, is a *De Morgan triple* and a triple $\langle S,T,n\rangle$ is called *Lukasiewicz triple*, if T is nilpotent.

Fuzzy implications obtained from Eq.(1.5) are usually referred to as *S-implications* (the symbol S is often used for denoting *t-conorms*) whereas fuzzy implications obtained from Eq.(1.6) are called *R-implications*, as they are closely connected with the so-called resituated semi group and fuzzy implications obtained from Eq.(1.7) are called *QL-implications*, since they were originally employed in quantum logic [35]. Also, in the literature, fuzzy implications obtained from Eq.(1.8) are called *Dishkant* implications or *D-implications* in short [38].

Contrary to the above *theoretical* definitions, in fuzzy systems applications the predominant practice (known as the *Mamdani* method) is to employ fuzzy *products* (symmetric operators, formally akin to t-norms) instead of implications, and to aggregate the results by union (usually by the $\max\{\ldots\}$). The fuzzy products most commonly used in applications are:

Mamdani rule $I_{\mathsf{M}}\left(a,b\right)=\min\left\{a,b\right\}$ and
Larsen rule $I_{\mathsf{La}}\left(a,b\right)=a\cdot b.$

Fuzzy products clearly do not reduce to the classical implication in the limit. In fact, they differ fundamentally from fuzzy implications in that they: a) abide by "*falsity implies nothing*" (rather than "*everything*") and b) do not distinguish between predicate (cause) and antecedent (effect).

In view of these properties, the Mamdani method is best suited to inference based on phenomenological information [35]. We refer to I_{M}, I_{La} as the "*engineering implications*", contrary to the fuzzy implications [39].

A number of basic properties of the classical (logic) implication has been generalized by fuzzy implications. Hence, a number of "*reasonable axioms*" emerged tentatively for fuzzy implications. Some of the aforementioned axioms are displayed next [35].

A1. $a \leq b \Rightarrow I(a,x) \geq I(b,x)$.	*Monotonicity in first argument*
A2. $a \leq b \Rightarrow I(x,a) \leq I(x,b)$.	*Monotonicity in second argument*
A3. $I(a, I(b,x)) = I(b, I(a,x))$.	*Exchange property*
A4. $I(a,b) = I(n(b), n(a))$.	*Contraposition*
A5. $I(1,b) = b$.	*Neutrality of truth*
A6. $I(0,a) = 1$.	*Dominance of falsity*
A7. $I(a,a) = 1$.	*Identity*
A8.$I(a,b) = 1 \Leftrightarrow a \leq b$.	*Boundary Condition*
A9. I is a continuous function.	*Continuity*

Fuzzy implications do not fulfill the above nine axioms. Fuzzy implications that satisfy all the listed axioms are characterized by the theorem of Smets and Magrez [46]. Note that all the S-implications fulfill the axioms A1, A2, A3, A5, A6 and, when the negation is strong, A4. Also, all the R-implications fulfill the axioms A1, A2, A5, A6 and A7. In generalI_{QL}-implications violate property A1, the conditions under which this property is satisfied fromI_{QL}-implications can be found in [18]. Note that it has been shown that: If *T* be a *t*-norm, *S* a *t*-conorm and n a strong negation, then the corresponding *QL*-operator, I_{QL}, is a *QL*-implication if and only if the corresponding *D*-operator, I_D, is a *D*-implication [38].

In addition, from the above definition of fuzzy implication, Fodor and Roubens [19] have given the following definition of fuzzy implications.

A fuzzy implication *I* is a function of the form: $I : [0,1] \times [0,1] \rightarrow [0,1]$, which hold $\forall a,b \in [0,1]$:

(i) $I(0,0) = I(0,1) = I(1,1) = 1$and $I(1,0) = 0$,

(ii) $a \leq b \Rightarrow I(a,x) \geq I(b,x)$,

(iii) $a \leq b \Rightarrow I(x,a) \leq I(x,b)$,

(iv) $I(0,a) = 1$,

(v) $I(a,1) = 1$,

(vi) $I(1,0) = 0$.

1.2.2 Metric Distance. Basic Notations and Definitions

A *metric distance* d in a set $X \neq \varnothing$, is a real function $d : X \times X \rightarrow \mathbb{R}_0^+$, which satisfies the following conditions:

(i) $\forall x, y \in X, d(x,y) \geq 0$,

(ii) $\forall x, y \in X, d(x,y) = 0 \Leftrightarrow x = y$, *Reflectivity*,

(iii) $\forall x, y \in X, d(x,y) = d(y,x)$, *Symmetry*,

(iv) $\forall x, y, z \in X, d(x,z) + d(z,y) \geq d(x,y)$, *Triangle Inequality*.

If d satisfies the above conditions (i), (iii), (iv) and also, satisfies $d(x,x)=0, \forall x \in X$ and $d(x,y)=0$ for some $x \neq y$, then it is a *pseudo-metric*. If d satisfies the conditions (i), (ii) and (iii), then it is a *semi-metric*.

Let d be a metric distance in a set $X \neq \varnothing$, then the following properties hold:

(i) $\forall x_1, x_2, ..., x_v \in X$, $d(x_1, x_v) \leq d(x_1, x_2) + d(x_2, x_3) + ... + d(x_{v-1}, x_v)$,

(ii) $\forall x, y, a \in X$, $|d(x,a) - d(y,a)| \leq d(x,y)$,

(iii) $\forall x, y, x', y' \in X$, $|d(x,y) - d(x',y')| \leq d(x,x') + d(y,y')$.

A pair (X,d), where X is a set it have a metric d, is called *metric space*.

Let a set $X \neq \varnothing$, an internal operation $+ : X \times X \to X$ and an external operation $\centerdot : \mathbb{R} \times X \to X$, the quadruple $(X, +, \mathbb{R}, \centerdot)$ is called *real linear space*, when hold the following:

(i) for the internal operation $+$:

$$\begin{aligned}
&\forall x, y, z \in X, x + (y + z) = (x + y) + z,\\
&\exists \theta \in X : \forall x \in X,\ \theta + x = x + \theta\ = x,\\
&\forall x \in X, \exists x' \in X, x + x' = x' + x = \theta,\\
&\forall x, y \in X, x + y = y + x.
\end{aligned}$$

(ii) for the external operation $\centerdot$:

$$\begin{aligned}
&\forall \kappa \in \mathbb{R}, \forall x, y \in X, \kappa \centerdot (x + y) = \kappa \centerdot x + \kappa \centerdot y,\\
&\forall \kappa, \lambda \in \mathbb{R}, \forall x \in X, (\kappa + \lambda) \centerdot x = \kappa \centerdot x + \ \lambda \centerdot x,\\
&\forall \kappa, \lambda \in \mathbb{R}, \forall x \in X, \kappa \centerdot (\lambda \centerdot x) = (\kappa \centerdot \lambda) \centerdot x,\\
&\forall x \in X, 1 \centerdot x = x.
\end{aligned}$$

Let X be a real linear space. A mapping $\|\| : X \to \mathbb{R}$ is called a *norm* of X, when satisfies the following three conditions:

(i) $\forall x \in X, \|x\| = 0 \Leftrightarrow x = 0$,

(ii) $\forall k \in \mathbb{R}, \forall x \in X, \|kx\| = |k| \cdot \|x\|$,

(iii) $\forall x, y \in X, \|x + y\| \leq \|x\| + \|y\|$.

Let $\|\|$ be a norm of X, then the following properties hold:

(i) $\forall x \in X, \|x\| = \|-x\|, (ii) \forall x \in X, \|x\| \geq 0$,

(ii) $\forall x, y \in X, \|x - y\| = \|y - x\|$,

(iii) $\forall x, y \in X, |\|x\| - \|y\|| \leq \|x \pm y\|$,

(iv) $\forall x_1, x_2, ..., x_\nu \in X, \|x_1 + x_2 + ... + x_\nu\| \leq \|x_1\| + \|x_2\| + ... + \|x_\nu\|$.

1.2.3 Metric Distances on Fuzzy Sets and Intuitionistic Fuzzy Sets

Various metric distances, involving fuzzy sets, have been proposed in [7, 15, 34]. Some common metrics, which are used for the description of the distance between fuzzy sets, are the following:

If the universe set X is finite, i.e. $X = \{x_1, ..., x_n\}$ then for any two fuzzy subsets A and B of X with membership functions $\mu_A(.)$ and $\mu_B(.)$, respectively, we have:

- *Hamming Distance*

$$d_H(A,B) = \sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)| \tag{1.9}$$

- *Normalized Hamming Distance*

$$d_{n-H}(A,B) = \frac{1}{n}\sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)| \tag{1.10}$$

- *Euclidean Distance*

$$d_E(A,B) = \sqrt{\sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2} \tag{1.11}$$

- *Normalized Euclidean Distance*

$$d_{n-E}(A,B) = \sqrt{\frac{1}{n}\sum_{i=1}^{n} (\mu_A(x_i) - \mu_B(x_i))^2} \tag{1.12}$$

Intuitionistic fuzzy sets (IFSs) have been proposed by Atanassov, in 1983, as a generalization mathematical framework of the traditional fuzzy sets (FSs) originated from an early work of Zadeh [1, 2, 3, 4, 5, 52, 53]. The main advantage of the IFSs is their property to cope with the hesitancy that may exist due to information impression. This is achieved by incorporating a second function, along with the membership function of the conventional FSs, called *non-membership function*. In this way, apart from the degree of the *belongingness*, IFSs also combine the notation of the *non-belongingness* to better describe the real status of the information.

We note that, Atanassov introduced the concept of the *intuitionistic fuzzy set*, or for short, as follows:

An *intuitionistic fuzzy set* A in X is an object of the following form:

$$A = \{\langle x, \mu_A(x), v_A(x)\rangle \,|x \in X\}$$

where the functions, $\mu_A : X \to [0,1]$ and $v_A : X \to [0,1]$, define the degree of membership and the degree of non-membership of the element $x \in X$, respectively and for every $x \in X : 0 \leq \mu_A(x) + v_A(x) \leq 1$.

If $\pi_A(x) = 1 - \mu_A(x) - v_A(x)$, then $\pi_A(x)$ is the degree of non-determinacy of the element $x \in X$ to the set A and $\pi_A(x) \in [0,1], \forall x \in X$.

It is easily seen that each fuzzy set is a particular case of the intuitionistic fuzzy set.

Also, if A is a fuzzy set then $\pi_A(x) = 0, \forall x \in X$.

As in the first introduction of the IFSs and in the consequent study on the fundamentals of the IFSs, a lot of attention has been paid on developing distance or similarity measures between the IFSs, as a way to apply them on several problems of the engineering life.

Atanassov suggested [5] the generalization of the above distances Eq.(1.9)–Eq.(1.12) for IFSs, as follows:

Let A, B be IFSs in X, with membership functions $\mu_A(.)$, $\mu_B(.)$ and with non-membership functions $v_A(.)$, $v_B(.)$, respectively, then:

- *Hamming Distance*

$$d_H(A,B) = \frac{1}{2}\sum_{i=1}^{n}\left[\left|\mu_A(x_i) - \mu_B(x_i)\right| + \left|\nu_A(x_i) - \nu_B(x_i)\right|\right] \tag{1.13}$$

- *Normalized Hamming Distance*

$$d_{n-H}(A,B) = \frac{1}{2n}\sum_{i=1}^{n}\left[\left|\mu_A(x_i) - \mu_B(x_i)\right| + \left|\nu_A(x_i) - \nu_B(x_i)\right|\right] \tag{1.14}$$

- *Euclidean Distance*

$$d_E(A,B) = \sqrt{\frac{1}{2}\sum_{i=1}^{n}\left[\left(\mu_A(x_i) - \mu_B(x_i)\right)^2 + \left(\nu_A(x_i) - \nu_B(x_i)\right)^2\right]} \tag{1.15}$$

- *Normalized Euclidean Distance*

$$d_{n-E}(A,B) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left[\left(\mu_A(x_i) - \mu_B(x_i)\right)^2 + \left(\nu_A(x_i) - \nu_B(x_i)\right)^2\right]} \tag{1.16}$$

1.3 Distance and Similarity Measures based on Fuzzy Implications

1.3.1 Distance Measures based on Fuzzy Implications and Matrices Norms

These measures are based on matrix norms and arise from the study of the aggregate plausibility of set-operations [7]. The definition and properties of each family member depend on a binary fuzzy operator and a matrix norm, both chosen by the user to match the metric to the application. Also, similarity measures provide the numerical tools for quality classification fuzzy binary operators to a specific problem.

The distance measure is defined as follows [7]:

Definition 1. Given two fuzzy sets A in $X = \{x_1, ..., x_m\}$ and B in $Y = \{y_1, ..., y_n\}$ with membership functions $\mu(\cdot)$ and $\nu(\cdot)$, and two binary fuzzy operators $I_1, I_2 \in \Im$, where $\Im$ be an available set of fuzzy implications, define a distance $d : \Im \times \Im \to [0, +\infty)$ based on some tensor- or operator-norm, $\|\cdot\|$, by:

$$d(I_1, I_2 : A, B) = \|\mathbf{\Sigma}_1(A, B) - \mathbf{\Sigma}_2(A, B)\|,$$

where $\mathbf{\Sigma}_k(A, B) = [I_k(\mu(x_i), v(y_j))]$, $1 \le i \le m$, $1 \le j \le n$, $k \in \{1, 2\}$, i.e., $\mathbf{\Sigma}_k(A, B) = [I_k(\mu(x_i), v(y_j))]$=

$$I_k\left(\begin{bmatrix} \mu(x_1) \\ \cdot \\ \cdot \\ \cdot \\ \mu(x_i) \\ \cdot \\ \cdot \\ \cdot \\ \mu(x_m) \end{bmatrix}, \begin{bmatrix} v(x_1), ..., v(x_j), ..., v(x_n) \end{bmatrix}\right) =$$

$$\begin{bmatrix} I(\mu(x_1), v(x_1)) & ... & I(\mu(x_1), v(x_j)) & ... & I(\mu(x_1), v(x_n)) \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ I(\mu(x_i), v(x_1)) & ... & I(\mu(x_i), v(x_j)) & ... & I(\mu(x_i), v(x_n)) \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ I(\mu(x_m), v(x_1)) & ... & I(\mu(x_m), v(x_j)) & ... & I(\mu(x_m), v(x_n)) \end{bmatrix}.$$

The best candidates for applications are norms of low order and akin to commonly used vector norms, which may arise either as tensor norms or as even operator norms. The three obvious choices are:

$$\begin{aligned} \|\Sigma\|_{\mathsf{nH}} &\triangleq \frac{\|\Sigma(A, B)\|_{\mathsf{H}}}{mn} \\ &\triangleq \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}|\sigma_{ij}|}{mn}, \end{aligned}$$

$$\begin{aligned} \|\Sigma\|_{\mathsf{nF}} &\triangleq \frac{\|\Sigma(A, B)\|_{\mathsf{F}}}{\sqrt{mn}} \\ &\triangleq \sqrt{\frac{\sum_{k=1}^{\min\{m,n\}} S_k^2(\Sigma)}{mn}}, \end{aligned}$$

$$\|\Sigma\|_{\mathsf{n2}} \triangleq \frac{\|\Sigma(A,B)\|_2}{\sqrt{mn}} \triangleq \frac{S_1(\Sigma)}{\sqrt{mn}} \equiv \sqrt{\frac{S_1^2(\Sigma)}{mn}}.$$

In direct analogy with the previous, the three practical choices for applications are:

$$\begin{aligned}\|\Sigma\|_{\mathsf{nH}} \to d_{\mathsf{nH}}(I_1, I_2 : A, B) &\triangleq \frac{\|\Sigma_1(A,B) - \Sigma_2(A,B)\|_{\mathsf{H}}}{mn} \\ &\triangleq \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}|I_{1,ij} - I_{2,ij}|}{mn},\end{aligned}$$

$$\begin{aligned}\|\Sigma\|_{\mathsf{nF}} \to d_{\mathsf{nF}}(I_1, I_2 : A, B) &\triangleq \frac{\|\Sigma_1(A,B) - \Sigma_2(A,B)\|_{\mathsf{F}}}{\sqrt{mn}} \\ &\triangleq \sqrt{\sum_{k=1}^{\min\{m,n\}} \frac{S_k^2(\Sigma_1 - \Sigma_2)}{mn}},\end{aligned}$$

$$\|\Sigma\|_{\mathsf{n2}} \to d_{\mathsf{n2}}(I_1, I_2 : A, B) \triangleq \frac{\|\Sigma_1(A,B) - \Sigma_2(A,B)\|_2}{\sqrt{mn}} \triangleq \frac{S_1(\Sigma_1 - \Sigma_2)}{\sqrt{mn}},$$

where $\{S_1, \ldots, S_{\min\{m,n\}}\}$ are the singular values of Σ in decreasing order. Note that, the singular values of a matrix are non-negative real numbers that characterize its essence as an operator (neglecting reflections and rotations, i.e., orthogonal transformations) [7].

Since all entries of $(\mathbf{\Sigma}_1 - \mathbf{\Sigma}_2)$ take values in $[-1, 1]$, all distances defined in terms of the norms proposed in the previous subsection satisfy (on any finite X, Y):

$$d(I_1, I_2 : A, B) \in [0, 1],$$

for all A, B in $X = \{x_1, ..., x_m\}$ and $Y = \{y_1, ..., y_n\}$, respectively, and $I_1, I_2 \in \Im$.

In the case of $I_1 = I_2$ the distance vanishes, regardless of the fuzzy sets and matrix norm chosen. Finally, if both fuzzy operators I_1 and I_2 are extensions of the same classical operator and A and B are classical sets, the matrices Σ_1 and Σ_2 are equal and the distance vanishes.

All these distance measures depend on the choice of fuzzy sets A and B. In principle, a metric $d(I_1, I_2 : X, Y)$ free of this dependence may be defined as the aggregation of $d(I_1, I_2 : A, B)$ over all possible pairs (A, B). In practice, aggregation by direct computation is not feasible.

So, a disadvantage of these measures is that, as they rely on the availability of two particular fuzzy sets, which, in the case of fuzzy implications, may be viewed as a set of predicates and a set of antecedents.

A family of distance measures for finite fuzzy sets is motivated by the corresponding family of distance measures for fuzzy implications proposed in above.

This measure formulates the information of each set in matrix structure, where matrix norms in conjunction with fuzzy implications can be applied to measure the distance between the fuzzy sets. The advantage of this novel distance measure is its flexibility, which permits different fuzzy implications to be incorporated by extending its applicability to several applications where the most appropriate implication is used.

For any finite universe set $X = \{x_1, ..., x_n\}$ the set of fuzzy sets on it is denoted by $FS(X)$. The distance measure, for finite fuzzy sets, is defined as follows [7]:

Definition 2. Consider two fuzzy sets $A, B \in FS(X)$, on a finite universe X such that $X \equiv \operatorname{supp} A \cup \operatorname{supp} B$, a fuzzy implication I and any tensor- or operator-norm $\|.\|$. Then

$$d(A, B : I) = \|\mathbf{\Sigma}(A, A) - \mathbf{\Sigma}(B, B)\|$$

defines a metric distance $d : FS(X) \times FS(X) \to [0, 1]$.

This definition actually introduces multiple families of metrics with different meanings, according to the fuzzy implication chosen. The family based on t-norms is akin to traditional normalized metrics on fuzzy sets (see Section1.3). The family based on fuzzy implications emphasizes plausibility distribution over scaling.

Both definitions are application-oriented, as they rely on the availability of two particular fuzzy sets, which, in the case of fuzzy implications, may be viewed as a set of predicates and a set of antecedents. Our definitions are also dependent on the choice of a matrix metric (i.e., either a tensor- or an operator-norm).

As always with fuzzy sets, one must choose the distance measure that best matches one's intuition of relative and absolute distances for the specific application at hand.

The above distance measure extended to intuitionistic fuzzy sets (IFSs). To point out that several researchers believe that, there is a strong connection between intuitionistic fuzzy sets, interval-valued fuzzy sets and L-fuzzy sets. So, these measures in intuitionistic fuzzy sets might be expressed equivalently using either intuitionistic fuzzy sets or interval-valued fuzzy sets. Hence, these measures can easily be extended more generally to lattice fuzzy (L-fuzzy) sets [9, 12, 14, 20, 21, 22, 48].

Let X denote a universe of discourse, where X is a finite and let $IFS(X)$ denote the set of all IFSs in X.

Definition 3. Given two IFSs,

$$A = \{\langle x, \mu_A(x), v_A(x)\rangle \,|x \in X\} \text{ and } B = \{\langle x, \mu_B(x), v_B(x)\rangle \,|x \in X\},$$

where $X = \{x_1, ..., x_n\}$ is a finite universe. Also, let I be a fuzzy implication and any tensor-or operator-norm $\|\cdot\|$. Then

$$d(A, B; I) \triangleq \|\mathbf{\Sigma}(\mu_A) - \mathbf{\Sigma}(\mu_B)\| + \|\mathbf{\Sigma}(v_A) - \mathbf{\Sigma}(v_B)\| \tag{1.17}$$

where $\mathbf{\Sigma}(\mu.) = \left[\underset{i=1,\ldots,n}{I}(\mu.(x_i), \mu.(x_i))\right]$, $\mathbf{\Sigma}(v.) = \left[\underset{i=1,\ldots,n}{I}(v.(x_i), v.(x_i))\right]$, defines a metric distance $d : IFS(X) \times IFS(X) \to [0, +\infty)$.

Note that, the $n \times n$ matrices $\mathbf{\Sigma}(\mu.)$ defined as follows:

$$\mathbf{\Sigma}(\mu.) \triangleq \left[\underset{i=1,\ldots,n}{I}(\mu.(x_i), \mu.(x_i))\right] =$$

$$I\left(\begin{bmatrix}\mu_{\cdot}(x_1)\\ \cdot\\ \cdot\\ \cdot\\ \mu_{\cdot}(x_n)\end{bmatrix},\begin{bmatrix}\mu_{\cdot}(x_1),...,\mu_{\cdot}(x_n)\end{bmatrix}\right)=$$

$$\begin{bmatrix} I(\mu_{\cdot}(x_1),\mu_{\cdot}(x_1)) & ... & ... & I(\mu_{\cdot}(x_1),\mu_{\cdot}(x_n))\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ I(\mu_{\cdot}(x_n),\mu_{\cdot}(x_1)) & ... & ... & I(\mu_{\cdot}(x_n),\mu_{\cdot}(x_n))\end{bmatrix}.$$

The above function $d(A,B;I)$ is a metric [7, 25, 26]. So, this definition actually introduces multiple families of metrics with different meanings, according to the binary operator chosen. In Eq.(1.17) the norm $\|\mathbf{\Sigma}\|$ is computed by using the largest non negative eigenvalue of the positive definite *Hermitian* matrix $\mathbf{\Sigma}^T\mathbf{\Sigma}(\mathbf{\Sigma}^T$ is the transpose of matrix $\mathbf{\Sigma})$ [6],

$$\|\mathbf{\Sigma}\| = \sqrt{\lambda_{\max}}.$$

Definition 4. The introduced distance is very flexible in the sense that it enables the usage of an appropriate fuzzy implication regarding the application, by resulting to a wide range of distances of different properties and capabilities. Also, the incorporation of different fuzzy implications combined with the matrix norms gives a wide range of distance measures with different properties and abilities.

1.3.2 Similarity Measures based on Fuzzy Implications and Matrices Norms

Similarity measures have become an important tool to quantify a similarity between two fuzzy sets. This topic has been discussed in many papers from different points of view [10, 11, 13, 27, 28, 29, 30, 31, 32, 33, 36, 37, 40, 44, 45, 51, 54].

Let X denote a finite universe of discourse and let two fuzzy sets, A, B, in X. Pappis and Karacapilidis [44] defined the *grade of similarity* $M(A,B)$ of the fuzzy sets A and B, by:

$$M(A,B)=\begin{cases}1 & \text{if} A=B=\varnothing,\\ \frac{\sum_{x\in X}\min(A(x),B(x))}{\sum_{x\in X}\max(A(x),B(x))} & \text{otherwise}.\end{cases}.$$

Two fuzzy sets and B in X, called *approximately equal* if and only if, given a small nonnegative number ε, it is $M(A,B)\le\varepsilon$.

The *grade of similarity* $M(A,B)$ of two fuzzy sets A and B in X, satisfy the following properties:

(i) $M(A,B)=M(B,A)$,

(ii) $A=B\Leftrightarrow M(A,B)=1$,

(iii) $\min(A,B)=0\Leftrightarrow M(A,B)=0$,

(iv) $M(A, A') = 1 \Leftrightarrow A = M*$,

(v) $M(A, A') = 0 \Leftrightarrow A = I$ or $A = \varnothing$,

where A' is the complement of A, while, I, $\varnothing$ and $M*$ denote the fuzzy sets, those with all grades of membership equal to 1, 0 or 0.5, respectively.

Also in [44] introduced the similarity measures, for two fuzzy sets A and B in a finite universe X:

$$S(A, B) = \begin{cases} 1 & \text{if} A = B = \varnothing, \\ 1 - \frac{\sum_{x \in X} |A(x) - B(x)|}{\sum_{x \in X} (A(x) + B(x))} & \text{otherwise.} \end{cases}$$

$$L(A, B) = 1 - \max_{x \in X} |A(x) - B(x)|.$$

When we consider the measure $L(A, B)$ for fuzzy sets A and B in arbitrary universes, we have to replace $\max$ by $\sup$.

Another way of defining similarity measures resulting from the generalization of the definition of equality of two (classical) sets to fuzzy sets. That is, the generalization of equivalence:

$$A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A).$$

In this case, as a degree of similarity is considered the degree of truth of the equality $A = B$, which is defined using a t-norm and a fuzzy implication. The degree of truth of $A \subseteq B$ can be defined using any fuzzy implication as $\bigcap_{x \in X} I(A(x), B(x))$.

This gives the following similarity measure, which is called “degree of sameness”, presented by Bandler and Kohout in [8]:

$$E(A, B) = \min \left(\inf_{x \in X} I(A(x), B(x)), \inf_{x \in X} I(B(x), A(x)) \right).$$

All distance measures for binary fuzzy operators introduced in Definition 1 are normalized and, therefore, give rise to corresponding similarity measures as follows:

The degree of similarity between two fuzzy implications I_1, I_2, as evaluated over a pair of fuzzy sets A and B, of the finite universes $X = \{x_1, ..., x_m\}$ and $Y = \{y_1, ..., y_n\}$, respectively, is defined as:

$$S(I_1, I_2 : A, B) = 1 - d(I_1, I_2 : A, B).$$

The similarity measures defined as above take values in $[0, 1]$, and are exactly 1 for $I_1 = I_2$. Also, these similarity measures S, satisfies the following properties:

(i) S *continuous*,

(ii) S *reflexive*,

(iii) S *symmetric*: $S(I_1, I_2 : A, B) = S(I_2, I_1 : A, B)$,

(iv) $0 \leq S(I_1, I_2 : A, B) \leq 1$,

(v) If $I_1 = I_2$ then $S(I_1, I_2 : A, B) = 1, \forall A, B$.

It is also worth noting that, in the case of the d_∞ norm, the corresponding similarity measure:

$$S_\infty (I_1, I_2 : A, B) = 1 - d_\infty (I_1, I_2 : A, B) = 1 - \max_i \max_j |I_1 (a_\iota, b_j) - I_2 (a_\iota, b_j)|,$$

is akin to the similarity measure for fuzzy sets $A, B \in FS(X)$ proposed in [7]:

$$L(A, B) = 1 - d_\infty (A, B).$$

Regarding the advantages, the numerical experiments with similarity measures on a set of fuzzy implications and engineering implications, and the equivalence classes resulting, showed that these metrics and the similarity measures are consistent with intuition. Consequently, these similarity measures provide the numerical means for the qualitative classification of fuzzy binary operands to a specific problem.

1.4 Distance and Similarity Measures based on Generalized Modus Ponens and Generalized Modus Tollens

1.4.1 Generalized Modus Ponens and Generalized Modus Tollens

Any unqualified conditional fuzzy proposition p of the form:

$$p: \text{ If } \chi \text{ is } A, \text{ then } \Upsilon \text{ is } B$$

is determined $\forall x \in X$ and $\forall y \in Y$ by the formula

$$R(x, y) = I(A(x), B(y)) \tag{1.18}$$

where I denotes a fuzzy implication and R expresses the relationship between the variables χ and Υ involved in the given proposition. The membership grade represents ($\forall x \in X$ and $\forall y \in Y$) the truth value of the proposition

$$p_{xy}: \text{ If } \chi = x, \text{ then } \Upsilon = y.$$

The truth values of propositions "$\chi = x$" and "$\Upsilon = y$" are expressed by the membership grades $A(x)$ and $B(y)$, respectively. Consequently, the truth values of proposition p_{xy}, given by $R(x, y)$, involve a fuzzy implication in which $A(x)$ is the truth value of the antecedent and $B(y)$ is the truth value of the consequent.

As known, contrary to classical logic, in fuzzy logic the meaning of fuzzy implication is not unique. So, the question "to select an appropriate fuzzy implication for approximate reasoning, for a particular situation", converted to "which fuzzy implication is most appropriate for calculating the fuzzy relation R" in the Eq.(1.18).

It is obvious that we can derive a criterion for the selection of fuzzy implications from each fuzzy inference rule that has a counterpart in classical logic. For example, from the *generalized modus ponens and generalized modus tollens*.

The Generalized Modus Ponens (GMP for short) has the form [35]:

$$\begin{array}{l} \text{Rule: if "}x\text{ is }A\text{", then "}y\text{ is }B\text{"} \\ \text{Fact: "}x\text{ is }A'\text{"} \\ \hline \text{Conclusion: "}y\text{ is }B'\text{"} \end{array}$$

where A, A' and B, B' are fuzzy sets of the universes X and Y, respectively and A', B' are the complements of A, B, respectively.

These fuzzy sets are not necessarily normalized. The rule GMP finds application in all fuzzy controllers. The GMP should coincide with the classical one in the special case when $A' = A$ and $B' = B$.

The classical modus ponens is the tautology: $(a \wedge (a \Rightarrow b)) \Rightarrow b$.

Modus ponens states that given two true propositions, "a" and "$a \Rightarrow b$", the truth of the proposition "b" may be inferred.

A criterion for selecting an appropriate fuzzy implication I may be derived from the requirement that the generalized modus ponens coincide with the classical modus ponens.

Criterion GMP: Any fuzzy implication suitable for approximate reasoning based on the GMP should satisfy the equation:

$$B(y) = \sup_{x \in X} T\left[A(x), I(A(x), B(y))\right].$$

Another criterion for selecting an appropriate fuzzy implication I may be derived from the requirement that the Generalized Modus Tollens (GMT for short), coincide with the classical modus tollens [35]. The GMT has the form:

$$\begin{array}{l} \text{Rule: if "}x\text{ is }A\text{", then "}y\text{ is }B\text{"} \\ \text{Fact: "}y\text{ is }B'\text{"} \\ \hline \text{Conclusion: "}x\text{ is }A'\text{"} \end{array}$$

The classical modus tollens is the tautology: $(\neg b \wedge (a \Rightarrow b)) \Rightarrow \neg a$.

Criterion GMT: Any fuzzy implication suitable for approximate reasoning based on the GMT should satisfy the equation:

$$n(A(x)) = \sup_{y \in Y} T\left[n(B(y)), I(A(x), B(y))\right], \forall x \in X.$$

Note that, some fuzzy implications are suitable for the generalized modus ponens and modus tollens. It does not mean, however, that these fuzzy implications are superior in general. Most papers investigate this problem by first choosing particular classes of conjunctions and implications and then testing whether the different requirements are fulfilled. There are lots of possible choices, but still not the "best" one [35, 39].

1.4.2 Distance Measures based on GMP and GMT

Let two finite fuzzy sets $A(x) : X \to [0,1]$, $B(y) : Y \to [0,1]$. We assume that $X = \{x_1, ..., x_M\}$ and $Y = \{y_1, ..., y_N\}$, respectively. Fuzzy set $A(x)$ describes the input and $B(y)$ describes the output of a fuzzy rule $A \to B$. We translate the fuzzy rule $A \to B$, to a fuzzy implication.

So, let $A = a_1/x_1 + a_2/x_2 + ... + a_n/x_n$ and $B = b_1/y_1 + b_2/y_2 + ... + b_n/y_n$ be two fuzzy subsets of $X = \{x_1, x_2, ..., x_n\}$ and $Y = \{y_1, y_2, ..., y_n\}$, respectively and let I be a fuzzy implication.

We construct the vector $[n(A_I(x_1)), ..., n(A_I(x_n))]$, as follows:

Let T be an appropriate t-norm such that the law of modus ponens hold, then:

$$[n(A_I(x_1)), ..., n(A_I(x_n))] = [n(B(y_1)), ..., n(B(y_n))] \circ I(A(x_i), B(y_j)),$$

where $\circ$ denote the compositional rule $\max - T$. The operation $\circ$ is equivalent to the product of two matrices while the multiplication is substituted with the operator T, where T is a continuous *t*-norm, and the aggregation with the operator $\max$.

Below two algorithms are given which select the "most appropriate" fuzzy implication, from two given fuzzy implications I_1, I_2 [23]. This aims to insert tools for the choice of a suitable fuzzy implication, from an existing set of fuzzy implications. In general, it is a difficult problem to select an appropriate fuzzy implication for approximate reasoning, for a particular situation.

The first algorithm calculate the distance between the vector "exit" $B(y_j)$ and the vector $B_I(y_i)$, which results from the *Criterion GMP*. The second algorithm calculate the distance between the vector $n(A(x_i))$ and the vector $n(A_I(x_i))$, which results from the *Criterion GMT*.

A1. Distance Measurement Algorithm based on GMP

Step 1: Calculate the $n \times n$ matrix$I(A(x_i), B(y_j))$.

Step 2: Calculate the $1 \times n$ matrix $[B_I(y_1), ..., B_I(y_n)]$, using the compositional rule $\max - T$, where T is a continuous *t*-norm.

Step 3: Finally, calculate the distance $d(B, B_I)$ between $B = [B(y_1), ..., B(y_n)]$ and $B_I = [B_I(y_1), ..., B_I(y_n)]$ using a metric distance between two fuzzy sets.

A2. Distance Measurement Algorithm based on GMT

Step 1: Calculate the $n \times n$ matrix $I(A(x_i), B(y_j))$.

Step 2: Calculate the $1 \times n$ matrix

$$[n(A_I(x_1)), ..., n(A_I(x_n))] = [n(B(y_1)), ..., n(B(y_n))] \circ I(A(x_i), B(y_j)),$$

using the compositional rule $\max - T$, where T is a continuous *t*-norm and n is a strong negation.

Step 3: Calculate the distance $d(n(A), n(A_I))$ between $[n(A(x_1)), ..., n(A(x_n))]$ and$[n(A_I(x_1)), ..., n(A_I(x_n))]$ using a metric distance between two fuzzy sets.

So, for every fuzzy implication I we get a pair $\langle d(B, B_I), d(n(A), n(A_I))\rangle$. The values of $d(B, B_I)$ and $d(n(A), n(A_I))$ depend from the degree that the fuzzy implication I satisfies the GMP and GMT, respectively.

1.4.3 Similarity Measures based on GMP and GMT

Similarly arises a similarity measure of fuzzy implications [42].

Let $A = a_1/x_1 + a_2/x_2 + ... + a_n/x_n$ and $B = b_1/y_1 + b_2/y_2 + ... + b_n/y_n$ be two fuzzy subsets of $X = \{x_1, x_2, ..., x_n\}$ and $Y = \{y_1, y_2, ..., y_n\}$, respectively. Let, also, I be a fuzzy implication. Define the matrix of $I(A(x_i), B(y_j))$ as follows:

$$I(A(x_i), B(y_j)) = \begin{bmatrix} I(a_1, b_1) & I(a_1, b_2) & ... & I(a_1, b_n) \\ I(a_2, b_1) & I(a_2, b_2) & ... & I(a_2, b_n) \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ I(a_n, b_1) & I(a_n, b_2) & ... & I(a_n, b_n) \end{bmatrix}, \ i, j = 1, ..., n.$$

We construct the vector $[B_I(y_1), B_I(y_2), ..., B_I(y_n)]$ as follows:

$$[B_I(y_1), B_I(y_2), ..., B_I(y_n)] = [A(x_1), A(x_2), ..., A(x_n)] \circ I(A(x_i), B(y_j)),$$

where $\circ$ denote, as above, the compositional rule $\max - T$ and T is a continuous *t*-norm.

To calculate the degree of similarity, between two fuzzy implications I_1 and I_2, based on GMP, as evaluated on a pair of fuzzy sets A, B of the finite universes $X = \{x_1, x_2, ..., x_n\}$ and $Y = \{y_1, y_2, ..., y_n\}$, respectively, follow the following algorithm:

A3. Similarity Measurement Algorithm based on GMP

Step 1: Calculate the $n \times n$ matrix $I_1(A(x_i), B(y_j))$.

Step 2: Calculate the $1 \times n$ matrix $B_{I_1}(y_i) = [B_{I_1}(y_1), ..., B_{I_1}(y_n)]$, using the compositional rule $\max - T$ where T is a continuous *t*-norm.

Step 3: Calculate $\|B_{I_1}(y_i)\|$, where $\|.\|$ denote a operator-norm on X.

Step 4: Calculate $\|B_{I_2}(y_i)\|$, repeating *Steps 1-3* for I_2.

Step 5: Finally, calculate $\min\left(\frac{\|B_{I_1}\|}{\|B_{I_2}\|}, \frac{\|B_{I_2}\|}{\|B_{I_1}\|}\right)$

So, the degree of similarity between two fuzzy implications I_1 and I_2 as evaluated over a pair of fuzzy sets A, B of the finite universes $X = \{x_1, x_2, ..., x_n\}$ and $Y = \{y_1, y_2, ..., y_n\}$, respectively, is defined as:

$$S(I_1, I_2 : A, B) = \min\left(\frac{\|B_{I_1}\|}{\|B_{I_2}\|}, \frac{\|B_{I_2}\|}{\|B_{I_1}\|}\right),$$

where, $\|\bullet\|$ denotes an operator-norm on X, and B_{I_1}, B_{I_2} are the vectors which are obtained from Step 3 and Step 4 of the *Algorithm A3*.

The above defined measure $S\left(I_1, I_2 : A, B\right)$, where I_1, I_2 are fuzzy implications and A,B finite fuzzy sets, is a similarity measure, because it satisfies the conditions:

(i) $0 \leq S\left(I_1, I_2 : A, B\right) \leq 1$,

(ii) $S\left(I_1, I_2 : A, B\right) = S\left(I_2, I_1 : A, B\right)$,

(iii) If $I_1 = I_2$ then $S\left(I_1, I_2 : A, B\right) = 1$.

1.5 Conclusions

A unified approach in constructing distance and similarity measures using fuzzy implications was presented in the previous sections. The proposed methodology exhibits a high degree of flexibility since it enables the incorporation of any fuzzy implication while it can be applied to any type of fuzzy sets (intuitionistic fuzzy sets, interval-valued fuzzy sets, etc.). The resulted measures satisfy the main properties of a metric and thus are theoretically correct, while their utility in specific applications constitutes the subject of future research.

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