CHAPTER 2

Toward a Synergy of a Lattice Implication Algebra with Fuzzy Lattice Reasoning – A Lattice Computing Approach

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Automated reasoning can be instrumental in real-world applications involving "intelligent" machines such as (semi-)autonomous vehicles as well as robots. From an analytical point of view, reasoning consists of a series of inferences or, equivalently, implications. In turn, an implication is a function which obtains values in a well-defined set. For instance, in classical Boolean logic an implication obtains values in the set $\{0, 1\}$, i.e. it is either true (1) or false (0); whereas, in narrow fuzzy logic an implication obtains values in the specific complete mathematical lattice unit-interval, symbolically $[0, 1]$, i.e. it is partially true/false. A lattice implication algebra (LIA)

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assumes implication values in a general complete mathematical lattice toward enhancing the representation of ambiguity in reasoning. This work introduces a LIA with implication values in a complete lattice of intervals on the real number axis. Since real numbers stem from real-world measurements, this work sets a ground for real-world applications of a LIA. We show that the aforementioned lattice of intervals includes all the enabling mathematical tools for fuzzy lattice reasoning (FLR). It follows a capacity to optimize, in principle, LIA-reasoning based on FLR as described in this work.

Keywords - Fuzzy lattice reasoning, lattice computing, lattice implication algebra.

2.1 Introduction

Reasoning [2, 15, 16, 17, 18, 45, 46, 47, 50, 51, 52] can be instrumental in real-world applications that involve uncertainty including inaccuracies and incomplete /ambiguous information. One of the main research directions of intelligent machine information processing regards processing of disparate types of information. In the aforementioned context, both the theory of uncertainty reasoning and its machine implementation are interesting. Classical automated reasoning, based on "certain" information, is dependable. To render dependable as well reasoning based on "non certain" information, it is necessary to establish a suitable algebraic logic system. Note that a number of logic systems such as many-valued logic (including fuzzy logic and lattice-valued logic), non monotonic logic, modal logic, probability logic and intuitionistic logic, all propose a logic basis (different from one another) for reasoning based on uncertain information [1, 3, 7, 10, 37, 38, 39, 49, 58, 62, 63, 65].

Uncertain information in a natural language is often represented in an ordered-structure [4, 5, 6, 8, 13]. In particular, usually, researchers use the elements in a totally-ordered set to represent uncertainty; for instance, in narrow fuzzy logic a number in the closed interval $[0, 1]$ is used. However, there is uncertain information, e.g. regarding incomparability, which is not totally-ordered. Therefore, in order to establish formal reasoning methods, novel methodologies should be sought in a suitable mathematical framework, namely mathematical lattice theory as detailed below.

This chapter is organized as follows. Section 2.2 details related work regarding logic and/or reasoning methods. Section 3 outlines basic lattice implication algebra (LIA) theory. Section 4 summarizes fuzzy lattice reasoning (FLR). Section 5 introduces a LIA on a complete lattice of Type-1 intervals amenable to FLR. Finally, section 6 concludes by summarizing our contribution and delineating future work.

2.2 Related Work

L.A. Zadeh introduced fuzzy logic in 1965 [71]. Ever since, scholars have focused mainly on fuzzy linguistic variables as well as on their application in intelligent information processing. The first publications in fuzzy set theory by Zadeh and Goguen demonstrated the intention of the authors to generalize the classical notion of a set as well as to accommodate fuzziness in the sense contained in a human language regarding judgment, evaluation and decision-making. More specifically, Zadeh writes: "The notion of a fuzzy set provides a convenient point of departure for the construction of a conceptual framework which parallels in many respects the framework used in the case of ordinary sets, but is more general than the latter and, potentially, may prove to have a much wider scope of applicability, particularly in the fields of pattern classification and information processing. Essentially, such a framework provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables", where "imprecision" here is meant as vagueness rather than as lack of knowledge about the precise value of a parameter (as in tolerance analysis). Fuzzy set theory provides a rigorous mathematical framework (there is nothing fuzzy about fuzzy set theory!) in which vagueness can be rigorously studied with precision. It can also be considered as a modeling language, well suited for situations in which fuzzy relations, criteria, and phenomena exist. The acceptance of this theory grew slowly in the 1960s and in the early 1970s. However, in the second half of the 1970s, the first successful practical applications of fuzzy set theory in automatic control regarding heating systems as well as cement factories boosted interest in this area. Furthermore, a successful application in washing machines /video cameras /cranes /subway trains, especially in Japan, boosted research in the 1980s. Note that around 4,000 publications existed in the year 1984, whereas in the year 2000 more than 30,000 publications had been reported. In particular, fuzzy set theory developed along the following two lines:

(a) On the one hand, in mathematics, fuzzy set theory has been enhanced by novel concepts ultimately including classical mathematical areas such as algebra, graph theory, topology, etc. by generalizing (i.e., fuzzifying) them [19, 20, 21, 22, 23, 24, 25, 36, 40, 41, 42, 43, 44, 48, 54, 56, 58, 59, 62, 63, 64, 65, 66, 67, 68, 72, 73]. This development is still ongoing.

(b) On the other hand, in applications, fuzzy technology for modeling, data mining etc. has been a viable alternative to classical modeling approaches.

In 1973, Zadeh proposed a definition for fuzzy controller thus paving the way for fuzzy control. In 1974, Mamdani presented a fuzzy controller for controlling a boiler as well as a steam engine. With the development of fuzzy control technology, researchers paid attention to basic theoretical problems including logic, reasoning, implication operators, etc. In the aforementioned context, the development of a theoretical basis for uncertain information processing became a necessity.

Many-valued logic, alternatively called multi-logic or multiple-valued logic, is a propositional calculus with more than two truth values. Multi-valued logic, that is an extension of classical two-valued logic, is an important research direction of non-classical logic. It has been argued that multi-valued logic can make machine decision-making amenable to human thinking. In classical, "Aristotelian" logic there are only two possible values, namely "true" and "false", for any proposition. Classical, two-valued logic may be extended to n-valued logic for $n > 2$. In particular, popular n-valued logics in the literature include the three-valued logic (developed by Lukasiewicz and Kleene, which assumes the values "true", "false", and "unknown"), the finite-valued (finitely-many valued) logic with more than three values, and the infinite-valued (infinitely-many valued) logic including fuzzy logic as well as probability logic. The first known logician who didn't fully accept the law of excluded middle was the Greek Aristotle (who, ironically, is also considered to be the first classical logician as well as the "father of logic"). Aristotle claimed that the laws of logic do not all apply to future events (De Interpretatione, ch. IX), but he did not propose any system of multi-valued logic to explain this isolated remark. Up to the 20th century, logicians followed Aristotelian logic, which assumes the law of the excluded middle. The idea of multi-valued logic was reconsidered in the 20th century. More specifically, the Polish logician and philosopher Jan Lukasiewicz proposed systems of many-valued logic in 1920, using a third value, namely "possible", to deal with Aristotle's paradox of the sea battle. Meanwhile, the American mathematician, Emil L. Post in 1921 independently proposed n additional degrees of truth for $n \geq 2$. Later, Jan Lukasiewicz and Alfred Tarski jointly formulated a logic with n truth values, where $n \geq 2$. Kurt Gödel in 1932 showed that intuitionistic logic is not finitely-many valued, furthermore he defined intermediate logic systems between classical logic and intuitionistic logic.

Many-valued logics constitute an interesting alternative to classical logic for modeling and reasoning. By allowing for additional truth values, they are able to represent uncertainty and/or disagreement. In the aforementioned context, a variety of applications has been considered in databases, knowledge representation, machine learning, circuit design and elsewhere. The first step in many-valued logic was taken by Lukasiewicz as well as by Post. More specifically, in 1920, Lukasiewicz introduced a three-valued logic system, where the third truth value could represent "neutral", "undefined" or an intermediate state. The latter was the first formal system in many-valued logic. Shortly later and independently, Post proposed a complete n-valued logic system (different from Lukasiewicz' logic system), which did not consider any underlying philosophical problems but rather it proposed a formal many-valued logic system. Afterwards, a number of many-valued logic systems was proposed with different philosophical backgrounds as well as with different application fields. Some representative works include: 3-valued logic systems introduced by a number of scholars; n-valued logic systems introduced mainly by Gödel (1930), Kalmar (1934), Sobocinski (1936), Slupecki (1938), Chang (1958), Rasiowa (1972), Cignoli (1982), Mangani (1973), Komori (1978), Di Zenzo (1986), De Glas (1989); logic systems with truth values in the closed interval $[0, 1]$ introduced mainly by Zadeh, Lee and Chang, Rasiowa, Gaines, Baldwin, Gottwald, Todt, Wu, Wang and other. In particular, in 1965, Zadeh introduced the concept "fuzzy set" which is of momentous significance in the study of non-classical logic. In 1967, Goguen extended the concept of fuzzy sets to L-fuzzy sets in which membership obtains values in a partially ordered set instead of in a totally ordered set such as $[0, 1]$.

Goguen studied lattice-valued logic and further proposed the first lattice-valued logic formal system based on complete lattice-ordered semigroups which initiated the study of lattice-valued logic. Thereafter, a number of researchers including Rosser, Turquette and Pavelka proposed further improvements on Goguen's logic system – Note that Pavelka's work is the most typical one. In 1979, Pavelka incorporated internal truth value in the language, established a fuzzy propositional logic system whose truth value set is an enriched residuated lattice and proved several important results regarding its axiomatizability; in particular, the followings axioms are most significant: (1) Any propositional logic with $L = [0, 1]$ and a continuous residuation operation (in particular, Lukasiewicz's implication) is axiomatizable, and (2) Any propositional logic in which L is a finite chain, is axiomatizable. Pavelka's work is concerned mainly with propositional fuzzy logic. Pavelka showed that the only natural way of formalizing fuzzy logic for truth values in the unit interval $[0, 1]$ is by the Lukasiewicz's implication operator or some isomorphic forms of it. Novak extended Pavelka's work to first-order fuzzy logic. Pavelka and Novak's work provided a relative general framework for lattice-valued logic system. Pavelka's as well as Novak's works have been influential in fuzzy logic. For instance, a number of authors including Turunen, Esteva, Godo, Hajek, Ying et al. have studied fuzzy logic in the light of Pavelka and Novak's work regarding many-valued logic based on a residuated lattice.

In the following we cite from the extensive literature regarding triangular norms and conorms as possible truth-function of conjunction and disjunction, and the induced truth-function for implication. Note that triangular norms were introduced in the framework of probabilistic metric spaces and they are applied in several fields including fuzzy set theory, fuzzy logics and their applications. Furthermore, residuated many-valued logics are related to continuous *t*-norms which are used as truth functions for the conjunction connective, and their residua as truth functions for the implication. Main examples are the Lukasiewicz, Gödel and Product logics, related to the Lukasiewicz *t*-norm $t(x, y) = \max\{0, x + y - 1\}$, Gödel *t*-norm $t(x, y) = \min\{x, y\}$ and Product *t*-norm $t(x, y) = xy$, respectively. Rose and Rosser presented completeness results for the Lukasiewicz logic, whereas Dummet presented results for the Gödel logic. Later, the aforementioned three authors axiomatized the Product logic. Hajek (1998) proposed the axiomatic system BL (i.e., Basic Logic) corresponding to a generic continuous *t*-norm, where the Lukasiewicz, Gödel and Product logics are special cases. Chang, Mundici, Belluce, Turunen, Hajek among others have studied many-valued logic based on MV (i.e., Multi-Valued) algebras. From the logical viewpoint, fuzzy logic can be regarded as a special case of many-valued or infinite-valued logic. A many-valued propositional logic with truth values in the unit interval $[0, 1]$ is often called fuzzy logic (in a narrow sense), where the conjunction is usually implemented by a triangular norm. There are different ways of implementing an implication. The latter (implementation) combined with the different implementations of a triangular norm results in a large collection of fuzzy logics with different semantics. For instance, Dubois and Prade investigated the possibilistic logic, whereas Liau et al. introduced a possibilistic residuated implication logic with applications for reasoning, where the semantics of the proposed logic is uniformly based on possibility theory. Each logic in the class is parameterized by a *t*-norm operation on $[0, 1]$, whereas the degree of implication between the possibilities of two formulas is parameterized explicitly by the residuated implication with respect to the *t*-norm. Furthermore, *t*-norm extensions to lattices have been considered [9, 14].

Another type of lattice-valued logic, namely quantum logic, emerged from the attempts of Jordan, et al. toward providing an axiomatic foundation for quantum mechanics. In particular, quantum logic was defined based on orthomodular lattices of closed convex subsets or closed linear subspaces in either a real or a complex Hilbert space; in either case the lattices are non-Boolean. Quantum logic is more general than Boolean logic since it does not require lattice distributivity. In fact, quantum logic does not even require modularity. The uncountably infinite orthonormal bases of a non-separable complex Hilbert space (H) can represent all conceivable experiments specifying a quantum mechanical system.

Boolean logic, fuzzy logic and quantum logic are typically studied in the context of mathematical lattice theory. In particular, first, Boolean lattices are typically used to study Boolean logic systems. Second, the narrow version of fuzzy logic developed by Zadeh (1965) is typically studied using the totally-ordered, complete lattice $[0, 1]$ of real numbers equipped with its associated distributive lattice structure; moreover, the general form of fuzzy logic allows for truth values in an arbitrary universal algebra. Third, quantum logics are typically constructed so that the truth values of propositions obtain values in an ortholattice.

Lattice-valued algebras of truth values have been used for theorem proving regarding both m-valued Post logics and algorithmic logics. For instance, Belnap has proposed a 4-valued logic that also uses the value "both" (i.e., "true and false"), to handle inconsistent assertions in database systems; Salzer has studied operators and distribution quantifiers in finite-valued logics based on semi-lattices; Hahnle has derived tableau-style axiomatizations of distribution quantifiers using Birkhoff's representation theorem for finite distributive lattices. Another type of lattice-value logic is based on a bilattice. The latter is a generalization of the classical two-valued logic; more specifically, a bilattice is a space of generalized truth values with two lattice orderings: One ordering represents the degree of truth, whatever that means, whereas the other ordering represents the degree of information or knowledge. Sofronie-Stokkermans has studied, first, a finite-valued logic whose truth values constitute a distributive lattice with certain type of operators and, second, automated theorem proving, where many results are based on the Priestley representation theorem of distributive lattices. In many cases the algebras of truth values associated to non-classical logic are of this type, such as SH n-logics. The latter are propositional finite-valued logics based on Symmetric Heyting algebras of order n, or SH n-logics for short; they focus on non-classical logics having as algebraic models bounded distributive lattices with certain types of operators. Automated theorem proving is also discussed. More specifically, the idea is to use a Priestley-style representation for distributive lattices with operators in order to define a class of Kripke-style models with respect to which the logic is both sound and complete. If this class of Kripke-style models is elementary, it can then be used for a translation to clause form. Satisfiability of the resulting clauses can be checked by resolution. Zadeh has proposed a logic system based on a linguistic truth values lattice. Liu has proposed a logic system based on a complete lattice with the dividing element. The implication connective in Liu's logic system is Kleene's implication. In addition, its truth values lattice contains a special element, namely dividing element, and some important results are given for $[0, 1]$. Wang has studied a logic system based on certain class of algebras and reported some important results.

Xu and colleagues have proposed lattice implication algebra, or LIA for short, by combining lattice and implication algebra in order to carry out uncertain information processing including incomparability in reasoning [68]. Furthermore, based on LIA, they have established lattice-valued propositional logic $LP(X)$ [63], lattice-valued first order logic $LF(X)$ [65] as well as fuzzy propositional logic $FP(X)$. In addition, based on work by Goguen, Pavelka and Novak they have established gradational L-type lattice-valued propositional logic system Lvpl [62] and first order logic system Lvfl [58] toward representing both the difference of degree among attributes of each intermediate-variation state of objects and the incomparability of objects. Uncertainty reasoning as well as resolution-based reasoning methods have been studied [53, 55, 56, 57, 59, 60, 61, 66, 67, 68, 69, 70] based on the aforementioned logic systems.

An alternative employment of mathematical lattice theory for fuzzy reasoning toward both learning and generalization in classification applications has been inspired from a study of the fuzzy adaptive resonance theory, or fuzzy-ART for short, neural network in the early 1990s [26]. We remark that fuzzy-ART pursues learning by computing lattice-ordered hyperboxes in the n-dimensional Euclidean space using two functions, namely Weber (choice) function and match function, which calculate the degree of inclusion of a hyperbox into another one. Later work [35] unified the aforementioned two functions of fuzzy-ART; hence, a new function, namely inclusion measure, emerged denoted by the Greek letter σ. Later, the applicability of an inclusion measure function was extended, first, to non-overlapping hyperboxes and, second, to a general mathematical lattice. In its extended capacity, a new name was sought instead of inclusion measure; hence, the name fuzzy order (function) was proposed [30]. Any employment of a fuzzy order function for decision-making is called fuzzy lattice reasoning, or FLR for short [29, 32].

This work paves the way toward a synergy of FLR with LIA by introducing a specific mathematical lattice that is the lattice of Type-1 intervals, amenable to both FLR and LIA. The proposed synergy is presented in the context of Lattice Computing, or LC for short – Recall that LC has been proposed as "an evolving collection of tools and mathematical modeling methodologies with the capacity to process lattice-ordered data per se including logic values, numbers, sets, symbols, graphs, etc." [33]. Note that preliminary descriptions of LC have been presented in [12, 27] as well as in [11], whereas a recent presentation of LC is described in [34]. Preliminary connections of FLR with lattice-valued logic have been reported elsewhere [28]. This work makes explicit connections between FLR with LIA using an improved mathematical notation.

2.3 Lattice Implication Algebra

In this chapter, we define a lattice implication algebra (LIA) by combining lattice and implication algebra and present some algebraic properties. Next, we introduce elementary notions as well as results from lattice-valued logic with truth-value in LIAs. The interested readers may refer to [68] for more details.

2.3.1 Definition, Examples and Basic Properties

Definition 1. Let $(L, \vee, \wedge, O, I)$ be a bounded lattice with an order-reversing involution $'$, greatest element I, least element O, and let

$$\rightarrow: L \times L \rightarrow L$$

be a mapping. Then $\mathcal{L} = (L, \vee, \wedge, ', \rightarrow, O, I)$ is called a lattice implication algebra, or LIA for short, if the following conditions hold for any $x, y, z \in L$:

(I_1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$;

(I_2) $x \to x = I$;
(I_3) $x \to y = y^{'} \to x^{'}$;
(I_4) If $x \to y = y \to x = I$ then $x = y$;
(I_5) $(x \to y) \to y = (y \to x) \to x$;
(L_1) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$;
(L_2) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$.

Example 1. Let $L = [0, 1]$ and let operators "$\vee, \wedge, ', \to$ " be defined as follows, for any $x, y, z \in L$:

$$\begin{aligned} x \vee y &= \max\{x, y\}, \\ x \wedge y &= \min\{x, y\}, \\ x' &= 1 - x, \text{ and} \\ x \to y &= \min\{x, 1 - x + y\}. \end{aligned}$$

Then $([0, 1], \vee, \wedge, ', \to, 0, 1)$ is a LIA, namely a Lukasiewicz implication algebra on $[0, 1]$.

Example 2. Let $L = \{a_i \mid i = 1, 2, ..., n\}$ and let operators "$\vee, \wedge, ', \to$" be defined as follows, for any $1 \le j, k \le n$:

$$\begin{aligned} a_j \vee a_k &= a_{\max\{i,k\}}, \\ a_j \wedge a_k &= a_{\min\{i,k\}}, \\ (a_j)^{'} &= a_{n-j+1}, \text{ and} \\ a_j \to a_k &= a_{\min\{n-j+k,n\}}. \end{aligned}$$

Then $(L, \vee, \wedge, ', \to, a_1, a_n)$ is a LIA, namely a Lukasiewicz implication algebra on a finite chain.

Example 3 (Boolean algebra)**.** Let $(L, \vee, \wedge, ', O, I)$ be a Boolean lattice. For any $x, y \in L$, define $x \to y = x' \vee y$. Then $(L, \vee, \wedge, ', \to, O, I)$ is a LIA.

From Example 3 it follows that we can define a LIA on any Boolean algebra.

Theorem 1. *Let $(L, \vee, \wedge, ', O, I)$ be a LIA. Then, $(L, \vee, \wedge)$ is distributive lattice.*

Theorem 2. *Let $(L, \vee, \wedge, ', O, I)$ be a LIA. Then, for any $x, y, z \in L$, the following statements hold:*

(1) If $I \to x = I$ then $x = I$;

(2) $I \to x = x$, $x \to O = x'$;

(3) $O \to x = I$, $x \to I = I$;

(4) $(x \to y) \to y = x \vee y$;

(5) $x \le y$ if and only if $x \to y = I$;

(6) $((x \to y) \to y) \to y = x \to y$;

(7) $(x \to y) \to x' = (y \to x) \to y'$;

(8) If $x \leq y$ then both $x \to z \geq y \to z$ and $z \to x \leq z \to y$;

(9) $(x \to y) \to ((y \to z) \to (x \to z)) = I$.

Theorem 3. *Let $(L, \vee, \wedge, ', O, I)$ be a LIA. Then, for any $x, y, z \in L$, the following statements hold:*

(1) $x \wedge y = ((x \to y) \to x')'$;

(2) $(x \to y) \to y = (y \to x) \to x$;

(3) $(x \to y) \to (z \to y) = z \to (x \vee y) = (y \to x) \to (z \to x) = (z \to x) \vee (z \to y)$;

(4) $(x \to y) \to (x \to z) = (x \wedge y) \to z = (y \to x) \to (y \to z) = (x \to z) \vee (y \to z)$;

(5) $(x \wedge z) \to (y \wedge z) = (x \wedge z) \to y$;

(6) $((x \to z) \wedge (z \to y)) \to (x \to y) = (x \to z) \vee (z \to y) \vee (x \to y)$;

(7) $(x \to (z \to y)) \to ((x \to z) \to (x \to y)) = x \vee z \vee ((x \to y) \vee (z \to y))$.

In a LIA $(L, \vee, \wedge, ', \to, O, I)$ two binary operations $\otimes$ and $\oplus$ are defined as follows, for any $x, y \in L$:

$$\begin{aligned} x \otimes y &= (x \to y')', \\ x \oplus y &= x' \to y. \end{aligned}$$

Theorem 4. *In a LIA $(L, \vee, \wedge, ', \to, O, I)$ for any $x, y, z \in L$, the following statements hold:*

(1) $x \otimes y = y \otimes x$, $x \oplus y = y \oplus x$;

(2) $x \otimes y \leq x \leq x \oplus y$, $x \otimes y \leq y \leq x \oplus y$;

(3) $x \to (x \otimes y) = x' \vee y = (x \oplus y) \to y$;

(4) $(x \to y) \otimes x = x \wedge y$;

(5) $x \to (y \to z) = (x \otimes y) \to z$;

(6) $x \leq y \to z$ if and only if $x \otimes y \leq z$;

(7) $x \otimes y = (x' \oplus y')'$, $x \oplus y = (x' \otimes y')'$.

Theorem 5. *If $(L, \vee, \wedge, ', \to, O, I)$ is a LIA then $(L, \otimes, \to, I)$ is a residuated lattice.*

Theorem 6. *Let L be a LIA.*

(1) In semigroup $(L, \oplus)$, the unique element with inverse is O.

(2) In semigroup $(L, \otimes)$, the unique element with inverse is I.

Corollary 1. *Suppose that L is a LIA and $|L| \geq 2$, then neither $(L, \oplus)$ nor $(L, \otimes)$ is a lattice-ordered group.*

Theorem 7. *Let L be a LIA, and $a, b \in L$. Then,*

$$a \otimes b = \wedge \{x \mid a \leq b \to x\}.$$

2.3.2 Filter Theory of Lattice Implication Algebras

Definition 2. Let $\mathcal{L} = (L, \vee, \wedge, ', \to, O, I)$ be a LIA and $F \subseteq L$. If

(1) $I \in F$, and

(2) for any $x, y \in L$, if $x \in F$, $x \to y \in F$ then $y \in F$, then F is called a filter of $\mathcal{L}$.

Theorem 8. *Let $\mathcal{L} = (L, \vee, \wedge, ', \to, O, I)$ be a LIA and $F \subseteq L$. Then, F is a filter of $\mathcal{L}$ if and only if*

(1) $I \in F$,

(2) for any $x, y \in F$, it follows that $x \otimes y \in F$, and

(3) for any $x, y \in L$ and $x \leq y$, if $x \in F$ then $y \in F$.

Remark 1. In Theorem 8 we can obtain (1) by (3). Therefore, it is easy to obtain the following Proposition.

Proposition 1. *Let $\mathcal{L} = (L, \vee, \wedge, ', \to, O, I)$ be a LIA and $F \subseteq L$. Then, F is a filter of $\mathcal{L}$ if and only if*

(1) for any $x, y \in F$, it follows $x \otimes y \in F$, and

(2) for any $x, y \in L$ and $x \leq y$, if $x \in F$ then $y \in F$.

Definition 3. Let $\mathcal{L} = (L, \vee, \wedge, ', \to, O, I)$ be a LIA and $F \subseteq L$. If F satisfies the following conditions:

(1) $I \in F$, and

(2) for any $x, y, z \in L$, $x \to (y \to z) \in F$ and $x \to y \in F$ imply $x \to z \in F$, then, F is called an implicative filter of $\mathcal{L}$.

Definition 4. Let $\mathcal{L} = (L, \vee, \wedge, ', \to, O, I)$ be a LIA and $F \subseteq L$. If F satisfies the following conditions:

(1) $I \in F$, and

(2) for any $x, y, z \in L$, $x \in F$ and $x \to ((y \to z) \to y) \in F$ imply $y \in F$, then, F is called a positive implicative filter of $\mathcal{L}$.

Theorem 9. *Let $\mathcal{L} = (L, \vee, \wedge,', \to, O, I)$ be a LIA and $F \subseteq L$. Then, F is filter of $\mathcal{L}$ if and only if*

(1) $I \in F$, and

(2) for any $x, y \in F$, $z \in L$, $(x \to (y \to z)) \to z \in F$.

Theorem 10. *Let $\mathcal{L} = (L, \vee, \wedge,', \to, O, I)$ be a LIA and $F \subseteq L$. Then, F is filter of $\mathcal{L}$ if and only if for any $x, y \in F$, $z \in L$, $x \to z \geq y$ implies $z \in F$.*

Proposition 2. *Let $\mathcal{L} = (L, \vee, \wedge,', \to, O, I)$ be a LIA and F is a positive implicative filter of $\mathcal{L}$. Then, for any $x \in L$, $x' \to x \in F$ implies $x \in F$.*

Theorem 11. *Let $\mathcal{L} = (L, \vee, \wedge,', \to, O, I)$ be a LIA and F be a filter of $\mathcal{L}$. Then, F is a positive implicative filter of $\mathcal{L}$ if and only if $(x' \to x) \to x \in F$, for any $x \in L$.*

Theorem 12. *Let $\mathcal{L} = (L, \vee, \wedge,', \to, O, I)$ be a LIA and F be a filter of $\mathcal{L}$. Then, the following conditions are equivalent:*

(1) F is a positive implicative filter of $\mathcal{L}$;

(2) for any $x, y \in L$, if $(x \to y) \to x \in F$ then $x \in F$;

(3) for any $x \in L$, $x \vee x' \in F$;

(4) for any $x, y, z \in L$, if $x \to (z' \to y) \in F$ and $y \to z \in F$ then $x \to z \in F$.

Remark 2. In a LIA $\mathcal{L}$, F is an implicative filter of $\mathcal{L}$ if and only if F is a positive implicative filter of $\mathcal{L}$.

Definition 5. Let $\mathcal{L}$ be a LIA and P be a proper filter of $\mathcal{L}$. Then, P is a prime filter of $\mathcal{L}$, if for any $x, y \in L$ and $x \vee y \in P$ we have either $x \in P$ or $y \in P$.

Definition 6. Let $\mathcal{L}$ be a LIA and P be a prime filter of $\mathcal{L}$. Then, P is a minimal prime filter of $\mathcal{L}$, if P is the minimal element of the partial order sets composed by all prime filters of $\mathcal{L}$.

Remark 3. Any prime filter must contain a minimal prime filter in a LIA. In fact, let P be a prime filter of $\mathcal{L}$ and let $\Sigma = \{Q \mid Q \text{ is a prime filter of} \mathcal{L} \text{and} Q \subseteq P\}$; obviously, $\Sigma \neq \emptyset$ and is a partially ordered set w.r.t. the conclusion relation of set. For any chain $\{Q_i\}_{i \in I}$ of Σ (I is an index set), it is straightforward to verify that $\bigcap_{i \in I} \{Q_i\}$ is a lower bound of this chain. Therefore, by Zorn Lemma, there is minimal element in Σ, which (minimal element) is the minimal prime filter contained in P.

Theorem 13. *Let $\mathcal{L}$ be a LIA and F be a filter of $\mathcal{L}$. Then, F is a prime filter of $\mathcal{L}$, if and only if F is a lattice prime filter of $\mathcal{L}$.*

Let F be a lattice filter of the LIA $\mathcal{L} = (L, \vee, \wedge,', \to, O, I)$. For any $a \in L$, put $\mathcal{F}_a = \{x \in L \mid x \to a \notin F\}$ and $\mathcal{K}(F) = \bigcap \mathcal{F}_a$.

Theorem 14. *Let $\mathcal{L} = (L, \vee, \wedge, ', \rightarrow, O, I)$ be a LIA. Then, P is minimal prime filter of $\mathcal{L}$ if and only if P is minimal prime lattice filter of $\mathcal{L}$.*

Theorem 15. *Let $\mathcal{L} = (L, \vee, \wedge, ', \rightarrow, O, I)$ be a LIA and P be a minimal prime filter of $\mathcal{L}$. Then, $P = \bigcup \{F_a \mid a \notin P\}$, where $F_a = \{x \in L \mid x \vee a = I\}$.*

Theorem 16. *Let $\mathcal{L} = (L, \vee, \wedge, ', \rightarrow, O, I)$ be a LIA and P be a minimal prime filter of $\mathcal{L}$. Then, $P = F_a\,(a \notin P)$ if and only if $\{F_b \mid b \notin P\}$ satisfies the maximal condition, where $F_b = \{x \in L \mid x \vee b = I\}$.*

2.4 Fuzzy Lattice Reasoning

In a mathematical lattice $(\mathsf{L}, \sqsubseteq)$, for any $x, y \in \mathsf{L}$, it is either $(x, y) \in \sqsubseteq$ or $(x, y) \notin \sqsubseteq$; in words, the relation "$x \sqsubseteq y$" is either true or false, respectively. However, in practical decision-making applications it is often useful to compute a degree of truth (in a narrow fuzzy logic sense) of the relation "$x \sqsubseteq y$". Therefore, we introduce the notion "fuzzy lattice" is order to fuzzify the (crisp) partial order relation in a lattice and extend it to every pair $(x, y) \in \mathsf{L} \times \mathsf{L}$ as follows.

Definition 7. A fuzzy lattice is a triple $(\mathsf{L}, \sqsubseteq, \mu)$, where $(\mathsf{L}, \sqsubseteq)$ is a (crisp) lattice and $(\mathsf{L} \times \mathsf{L}, \mu)$ is a conventional (or, equivalently, narrow) fuzzy set such that $\mu(x, y) = 1 \Leftrightarrow x \sqsubseteq y$.

We pursue a fuzzy lattice based on a "fuzzy order" function defined next.

Definition 8. Let $(\mathsf{L}, \sqsubseteq)$ be a lattice. A function $\sigma : \mathsf{L} \times \mathsf{L} \rightarrow [0, 1]$ is called fuzzy order if and only if the following two properties hold.
C1. $u \sqsubseteq w \Leftrightarrow \sigma(u, w) = 1$,
C2. $u \sqsubseteq w \Rightarrow \sigma(x, u) \leq \sigma(x, w)$ (Consistency).

We remark that the notations $\sigma(u, w)$ and $\sigma(u \sqsubseteq w)$ are used interchangeably. Any employment of a fuzzy order function, e.g., in decision-making applications, is called fuzzy lattice reasoning, or FLR for short. We point out that a fuzzy order function σ supports two different modes of reasoning, namely Generalized Modus Ponens and Reasoning by Analogy [29], as explained in the following. On the one hand, Generalized Modus Ponens is implemented given both a rule $a \rightarrow c$ and an antecedent $x \sqsubseteq a$; it follows $x \rightarrow c$. On the other hand, Reasoning by Analogy is implemented given both a set of rules $a_i \rightarrow c_i$, $i \in \{1, \ldots, L\}$ and an antecedent x such that $x \not\sqsubseteq a_i$, $i \in \{1, \ldots, L\}$. Then we conclude c_{J} such that $\mathsf{J} \triangleq \arg\max_{i \in \{1, \ldots, L\}} \sigma(x \sqsubseteq a_i)$.

An order function can be defined in a general lattice $(\mathsf{L}, \sqsubseteq)$ based on a positive valuation (real) function $\nu : \mathsf{L} \rightarrow \mathsf{R}$ as explained below – Recall that a valuation is a real function $\nu : \mathsf{L} \rightarrow \mathsf{R}$ that satisfies $\nu(x) + \nu(y) = \nu(x \sqcap y) + \nu(x \sqcup y)$; a valuation function is called monotone if and only if $x \sqsubseteq y \Rightarrow \nu(x) \leq \nu(y)$, whereas it is called positive if and only if $x \sqsubset y \Rightarrow \nu(x) < \nu(y)$.

In practice, we often deal with a complete lattice $(\mathsf{L}, \sqsubseteq)$ with least and greatest elements o and i, respectively; furthermore, due to the so-called "reasonable constraints" (i) $\nu(o) = 0$ and (ii) $\nu(i) < +\infty$, our interest here focuses on non-negative positive valuation functions $\nu : \mathsf{L} \rightarrow \mathsf{R}_0^+$. In conclusion, given a (non-negative) positive valuation function $\nu : \mathsf{L} \to \mathsf{R}_0^+$ in a lattice $(\mathsf{L}, \sqsubseteq)$, it turns out that both functions sigma-join given by $\sigma_\sqcup(x,u) = \frac{\nu(u)}{\nu(x \sqcup u)}$, and sigma-meet given by $\sigma_\sqcap(x,u) = \frac{\nu(x \sqcap u)}{\nu(x)}$, are fuzzy orders.

Let $(\mathsf{L}, \sqsubseteq)$ be a lattice with a fuzzy order $\sigma : \mathsf{L} \times \mathsf{L} \to [0,1]$; then, the triple $(\mathsf{L}, \sqsubseteq, \sigma)$ is a fuzzy lattice. That is, a fuzzy order (σ) in $(\mathsf{L}, \sqsubseteq)$ fuzzifies a crisp lattice $(\mathsf{L}, \sqsubseteq)$ by transforming it to a fuzzy lattice; therefore, it enables the quantified comparison of any two elements in $(\mathsf{L}, \sqsubseteq)$.

Our interest here focuses on the partially ordered set $(\mathsf{I}, \sqsubseteq)$ of conventional intervals in a lattice $(\mathsf{L}, \sqsubseteq)$, i.e.$I = \{[a,b] | a, b \in L \,\text{and}\, a \sqsubseteq b\}$. It turns out that $(\mathsf{I} \cup \{\emptyset\}, \subseteq)$ is a lattice, where the meet $(\cap)$ is defined as (1) $x \cap \emptyset = \emptyset$, $\forall x \in (\mathsf{I} \cup \{\emptyset\})$ and (2) $[a,b] \cap [c,d] = [a \sqcup c, b \sqcap d]$ if $a \sqcup c \sqsubseteq b \sqcap d$ and $[a,b] \cap [c,d] = \emptyset$ if $a \sqcup c \not\sqsubseteq b \sqcap d$, for $[a,b], [c,d] \in \mathsf{I}$; whereas, the join $\left(\dot{\cup}\right)$ is defined as (1) $x \dot{\cup} \emptyset = x$, $\forall x \in (\mathsf{I} \cup \{\emptyset\})$ and (2) $[a,b] \dot{\cup} [c,d] = [a \sqcap c, b \sqcup d]$, for $[a,b], [c,d] \in \mathsf{I}$.

Consider a complete lattice $(\mathsf{L}, \sqsubseteq)$ with least and greatest elements o and i, respectively. We represent the least element of lattice $(\mathsf{I}_1 = \mathsf{I} \cup \{\emptyset\}, \sqsubseteq)$ as $O = [i, o] = \emptyset$. The corresponding order relation $[a,b] \sqsubseteq [c,e]$ is defined as $[a,b] \sqsubseteq [c,e] \Leftrightarrow$ "$c \sqsubseteq a.AND.b \sqsubseteq e$". An advantage of the representation $O = [i,o]$ is that the latter order relation is compatible with the equivalence $[a,b] \subseteq [c,e] \Leftrightarrow c \sqsubseteq a \sqsubseteq b \sqsubseteq e$ in the partially ordered set $(\mathsf{I}, \sqsubseteq)$ of conventional intervals. Another advantage of the representation $O = [i,o]$ is that the definition of both the meet and the join in lattice $(\mathsf{I}_1, \sqsubseteq)$ is in accordance with the corresponding definitions in lattice $(\mathsf{I} \cup \{\emptyset\}, \subseteq)$, because the meet $(\sqcap)$ in lattice $(\mathsf{I}_1, \sqsubseteq)$ is computed by $[a,b] \sqcap [c,e] = [a \sqcup c, b \sqcap e]$ if $a \sqcup c \sqsubseteq b \sqcap e$ and $[a,b] \sqcap [c,e] = [i,o]$ if $a \sqcup c \not\sqsubseteq b \sqcap e$, whereas the join $(\sqcup)$ in $(\mathsf{I}_1, \sqsubseteq)$ is computed by $[a,b] \sqcup [c,e] = [a \sqcap c, b \sqcup e]$. An element of the set $\mathsf{I}_1 = \mathsf{I} \cup \{[i,o]\}$ is called Type-1 (T1) interval. In strict mathematical terms we say that the lattices $(\mathsf{I} \cup \{\emptyset\}, \sqsubseteq)$ and $(\mathsf{I}_1, \sqsubseteq)$ are isomorphic, symbolically $(\mathsf{I} \cup \{\emptyset\}, \subseteq) \approx (\mathsf{I}_1, \sqsubseteq)$.

Our intention is to define a fuzzy order function in lattice $(\mathsf{I}_1, \sqsubseteq)$ based on positive valuation function $\nu : \mathsf{L} \to \mathsf{R}_0^+$ in lattice $(\mathsf{L}, \sqsubseteq)$ as explained in the following. Assume a dual isomorphic function $\theta : (\mathsf{L}, \sqsubseteq)^\partial \to (\mathsf{L}, \sqsubseteq)$. It turns out that (1) function $\nu_1 : \mathsf{L} \to \mathsf{R}_0^+$ given by $\nu_1(a,b) = \nu(\theta(a)) + \nu(b)$ is a positive valuation function in lattice $(\mathsf{L}, \sqsubseteq)^\partial \times (\mathsf{L}, \sqsubseteq) = \left(\mathsf{L}, \sqsubseteq^\partial\right) \times (\mathsf{L}, \sqsubseteq) = (\mathsf{L} \times \mathsf{L}, \sqsupseteq \times \sqsubseteq)$, and (2) lattice $(\mathsf{I}_1, \sqsubseteq)$ is order embedded in lattice $(\mathsf{L} \times \mathsf{L}, \sqsupseteq \times \sqsubseteq)$.

Two fuzzy order functions can be defined in lattice $(\mathsf{L} \times \mathsf{L}, \sqsupseteq \times \sqsubseteq)$ as follows:

$$\sigma_\sqcap([a,b],[c,e]) = \begin{cases} 1, & \text{for } [a,b] = \emptyset \\ \frac{\nu(\theta(a \sqcup c)) + \nu(b \sqcap e)}{\nu(\theta(a)) + \nu(b)}, & \text{for } [a,b] \supset \emptyset \end{cases}, \text{ and}$$

$$\sigma_\sqcup([a,b],[c,e]) = \begin{cases} 1, & \text{for } [a,b] \sqcup [c,e] = \emptyset \\ \frac{\nu(\theta(c)) + \nu(e)}{\nu(\theta(a \sqcap c)) + \nu(b \sqcup e)}, & \text{for } [a,b] \sqcup [c,e] \supset \emptyset \end{cases}.$$

Since lattice $(\mathsf{I}_1, \sqsubseteq)$ is embedded in lattice $(\mathsf{L} \times \mathsf{L}, \sqsupseteq \times \sqsubseteq)$ both aforementioned functions sigma-meet $\sigma_\sqcap(.,.)$ and sigma-join $\sigma_\sqcup(.,.)$ are also valid fuzzy orders in lattice $(\mathsf{I}_1, \sqsubseteq)$.

The previous theory can be extended to the power-set 2^L of a lattice $(\mathsf{L}, \sqsubseteq)$ as explained in the following [31]. In particular, we define a partial order $\sqsubseteq$ in 2^L such that for $U, W \in 2^{\mathsf{L}}$, where $U = \{u_1, \ldots, u_I\}$ and $W = \{w_1, \ldots, w_J\}$, it is $U \sqsubseteq W \Leftrightarrow (\forall i \in \{1, ..., I\}, \exists j \in \{1, ..., J\} \mid u_i \sqsubseteq w_j)$. In conclusion, the lattice $\left(2^{\mathsf{L}}, \sqsubseteq\right)$ emerges with $U \sqcap W = \bigcup\limits_{i,j} \{u_i \sqcap w_j\}$ and $U \sqcup W = \bigcup\limits_{i,j} \{u_i \sqcup w_j\}$.

A subset S of L is called simplified if S includes, exclusively, incomparable elements of L. Let $\pi\left(2^{\mathsf{L}}\right)$ be the subset of the power-set 2^{L} including all the simplified subsets of L and only those. Our interest here focuses on the lattice $\left(\pi\left(2^{\mathsf{L}}\right), \sqsubseteq\right)$ because every element of $\pi\left(2^{\mathsf{L}}\right)$ maximizes a fuzzy order function due to the Consistency property (see in Definition 8). In conclusion, we define a fuzzy order function σ_c : $\pi\left(2^{\mathsf{L}}\right) \times \pi\left(2^{\mathsf{L}}\right) \to [0, 1]$ as follows.

Theorem 17. *Let function $\sigma : \mathsf{L} \times \mathsf{L} \to [0, 1]$ be a fuzzy order in a lattice $(\mathsf{L}, \sqsubseteq)$. Then, function $\sigma_c : \pi\left(2^{\mathsf{L}}\right) \times \pi\left(2^{\mathsf{L}}\right) \to [0, 1]$ given by the convex combination $\sigma_c (U \sqsubseteq W) = \sum_{i=1}^{I} \lambda_i \max\limits_{j \in \{1,...,J\}} \sigma(u_i \sqsubseteq w_j)$ is a fuzzy order.*

We remark that by "convex combination" in Theorem 17 we mean a set $\{\lambda_1, \ldots, \lambda_{\mathsf{I}}\}$ of positive numbers such that $\lambda_1 + \ldots + \lambda_{\mathsf{I}} = 1$.

2.5 A Lattice Implication Algebra Amenable to Fuzzy Lattice Reasoning

In this section we apply the general lattice theory presented in the previous section to the complete lattice $\left(\bar{\mathsf{R}}, \leq\right)$ of extended real numbers, i.e. $\bar{\mathsf{R}} = \cup \{+\infty, -\infty\}$, with least and greatest elements $-\infty$ and $+\infty$, respectively.

Any strictly increasing function $\nu : \bar{\mathsf{R}} \to \mathsf{R}_0^+$ is a positive valuation function on $\left(\bar{\mathsf{R}}, \leq\right)$. In particular, our interest is in positive valuation functions that satisfy the aforementioned "reasonable constraints" (i) $\nu(-\infty) = 0$ and (ii) $\nu(+\infty) < +\infty$. Moreover, any strictly decreasing function is an eligible dual isomorphic function θ : $\left(\bar{\mathsf{R}}, \leq\right)^{\partial} \to \left(\bar{\mathsf{R}}, \leq\right)$. In particular, our interest is in dual isomorphic functions such that both $\theta(-\infty) = +\infty$ and $\theta(+\infty) = -\infty$ in order to utilize the whole domain of the corresponding positive valuation function $\nu : \bar{\mathsf{R}} \to R_0^+$.

Furthermore, our interest is in the complete lattice $(\mathsf{I}_1, \sqsubseteq) \approx (\mathsf{I} \cup \{\emptyset\}, \subseteq)$ of Type-1 intervals with least and greatest elements denoted by $O = [+\infty, -\infty] = \emptyset$ and $I = [-\infty, +\infty] = \bar{\mathsf{R}}$, respectively. The corresponding partial order is defined as $[a, b] \subseteq [c, d] \Leftrightarrow (c \leq a.AND.b \leq d)$. The supremum operation, denoted by $\dot{\bigcup}$, is computed by $[a, b] \dot{\bigcup} [c, d] = [a \wedge c, b \vee d]$; whereas, the infimum operation, that is the set intersection $\bigcap$, is computed by $[a, b] \bigcap [c, d] = [a \vee c, b \wedge d]$ if and only if $a \vee c \leq b \wedge d$, otherwise $[a, b] \bigcap [c, d] = \emptyset$.

In the following we will consider a measure space. The latter (measure space) is defined as a triple $(X, \Sigma_X, m_{\Sigma_X})$, where X is a set, Σ_X is a σ-algebra over X, and m_{Σ_X} is a measure function over Σ_X – Recall that a σ-algebra over a set X is a collection of subsets of X that satisfies: (1) $\emptyset \in \Sigma_X$, (2) $A \in \Sigma_X$ implies $(X - A) \in \Sigma_X$, and (3) for a collection of sets $A_i \in \Sigma_X$ indexed by a countable indexing set D it follows $\left(\bigcup_{i \in D} A_i\right) \in \Sigma_X$. In words, a σ-algebra includes the empty set and it is closed under both complementation and countable unions. We remark that a measure is a set function $m_{\Sigma_X} : \Sigma_X \to \mathsf{R}_0^+$, which assigns a size to every "measurable set" element in Σ_X. Note that a measure m_{Σ_X} is required, by definition, to satisfy (1) $m_{\Sigma_X}(\emptyset) = 0$, and (2) for any countable indexing set D, and any collection of pairwise disjoint sets $A_i \in \Sigma_X$ indexed by $i \in D$, it holds $m_{\Sigma_X}\left(\bigcup_{i\in D} A_i\right) = \sum_{i\in D} m_{\Sigma_X}(A_i)$.

In this work we assume a measure space $(X, \Sigma_X, m_{\Sigma_X})$ such that $X = \bar{\mathsf{R}}$, $\Sigma_X = \mathsf{I} \cup \{\emptyset\}$, furthermore a measure m_{Σ_X} over Σ_X is defined as $m_{\Sigma_X}([a, b]) = \nu(b) - \nu(a)$, where $\nu : \bar{\mathsf{R}} \to \mathsf{R}_0^+$ is a (strictly increasing) positive valuation function.

Let 2^{I} be the power-set of I. Consider a simplified subset $s(2^{\mathsf{I}})$ of 2^{I} defined as follows: $s(2^{\mathsf{I}}) = \{\{\delta_i\} \mid \delta_i \in \mathsf{I}, i \in D$, where D is an index set, such that (a) $m(\delta_i) > 0$ and (b) $\delta_i \cap \delta_j = \emptyset$ for $i, j \in D$ with $i \neq j\} \cup \{\emptyset\}$. In words, an element of $s(2^{\mathsf{I}})$, other than the empty set, is a set of intervals $s = \{..., \delta_i, ..., \delta_j, ...\}$ such that (a) any interval δ_i in s has a non-zero measure $m(\delta_i)$, and (b) not two intervals δ_i and δ_j intersect. In other words, in a (subset) element of $s(2^{\mathsf{I}})$ (a) trivial intervals $[a, a]$ are excluded, and (b) two overlapping intervals, say $[a, b]$ and $[c, d]$, are replaced by their least upper bound interval $[a \wedge c, b \vee d]$. For example, even though the set $R = \{[1, 2], [3, 5], [4, 7]\}$ belongs to the power-set 2^{I_1}, it does not belong to $s(2^{\mathsf{I}})$ because $[3, 5] \sqcap [4, 7] = [3 \vee 4, 5 \wedge 7] = [4, 5] \neq \emptyset$; therefore, the set $\{[1, 2], [3, 5], [4, 7]\}$ is replaced by the set $\{[1, 2], [3, 7]\}$.

The set $s(2^{\mathsf{I}})$ is a lattice denoted by $\left(s(2^{\mathsf{I}}), \sqsubseteq\right)$. In particular, lattice $\left(s(2^{\mathsf{I}}), \sqsubseteq\right)$ is complete with greatest element $I = \{[-\infty, +\infty]\}$ and least element $O = \emptyset$. The join in lattice $\left(s(2^{\mathsf{I}}), \sqsubseteq\right)$ is defined as $R \sqcup S = \bigcup\limits_{i,j} \{r_i \cup s_j\} \in s(2^{\mathsf{I}})$. For example,

(a) $\{[1, 2], [3, 5]\} \sqcup \{[2, 3]\} = \{[1, 5]\}$;

(b) $\{[1, 3], [5, 8]\} \sqcup \{[0, 1], [4, 7]\} = \{[0, 3], [4, 8]\}$.

The meet in lattice $\left(s(2^{\mathsf{I}}), \sqsubseteq\right)$ is defined as $R \sqcap S = \bigcup \{r_i \cap s_j\}$. For example,

(a) $\{[1, 2], [3, 5]\} \sqcap \{[2, 3]\} = \{[2, 2], [3, 3]\} = \emptyset$;

(b) $\{[1, 3], [5, 8]\} \sqcap \{[0, 1], [4, 7]\} = \{[1, 1], [5, 7]\} = \{[5, 7]\}$.

For $R \in s(2^{\mathsf{I}})$, the negation R' is defined as the $R' = [-\infty, +\infty] \setminus R$. For example,

(a) $\{[2, 3]\}' = \{[-\infty, 2], [3, +\infty]\}$;

(b) $\{[-1, 0], [1, 2]\}' = \{[-\infty, -1], [0, 1], [2, +\infty]\}$.

It turns out that $\left(s(2^{\mathsf{I}}), \sqsubseteq, \sqcup, \sqcap\right)$ is a distributive lattice. Furthermore, $\left(s(2^{\mathsf{I}}), \sqcup, \sqcap, ', O, I\right)$ is a Boolean algebra. We define an implication $(\to)$ as follows: $R \to S = R' \sqcup S$. In conclusion, by Example 3, it follows that $\mathcal{L} = \left(s\left(2^{\mathsf{I}}\right), \sqcup, \sqcap, ', \to, O, I\right)$ is a LIA.

2.6 Discussion and Conclusion

This work has paved the way toward a synergy of lattice implication algebra (LIA) and fuzzy lattice reasoning (FLR) via the partially ordered set $(\mathsf{I}, \subseteq)$ of conventional intervals on the real axis. In conclusion, this work has shown that the simplified subset $s\left(2^{\mathsf{I}}\right) \subseteq 2^{\mathsf{I}}$ defines a LIA; furthermore $s\left(2^{\mathsf{I}}\right)$ is amenable to FLR.

The practical advantage of $(\mathsf{I}, \subseteq)$ is that it ubiquitously emerges from measurements in practice. Hence, using FLR it is possible to optimize LIA-reasoning, for example by choosing the best truth value. Nevertheless, before using LIA $\mathcal{L} = \left(s\left(2^{\mathsf{I}}\right), \sqcup, \sqcap, ', \rightarrow, O, I\right)$ for reasoning we might need to view the world from a novel perspective as delineated next.

An element of lattice $\left(s\left(2^{\mathsf{I}}\right), \sqsubseteq\right)$, that is a set of intervals, has to be treated as a "truth value" by a LIA. In other words, we need to treat measurements as truth values. For example, suppose that a temperature measurement is in the set $\{[16, 18], [19, 20]\}$ in centigrade (^{o}C) temperate measurement unit; then, the set $\{[16, 18], [19, 20]\}$ has to be treated as a LIA truth value.

Future work plans include theoretical extensions driven by practical applications. In particular, on the one hand, regarding theory, extensions of this work might also be sought in the mathematical lattice of fuzzy numbers based on their (interval) a-cuts; likewise extensions can be sought in the mathematical lattice of Intervals' Numbers, or INs for short [34]. On the other hand, regarding practical applications, machine intelligence might be pursued, e.g. in mobile /humanoid robot applications, toward planning by reasoning that accommodates uncertainty.

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